JOURNAL OF APPROXIMATION THEORY 56, 241–244 (1989)

# Fixed Point Theorems and Invariant Approximations

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*Communicated by G. Meinardus*

Received July 16, 1984; revised July 30, 1987

In [2] Meinardus introduced the notion of invariant approximation. The theorem of Meinardus was generalized by Subrahmanyam in [6] and Smoluk in [4]. The object of the present paper is a genralization of Dotson's fixed point theorem and its application to invariant approximation. We obtain a theorem which generalizes both of the above-mentioned results.

## I. PRELIMINARIES

If $E$ is a linear space, a subset $S$ of $E$ is said to be star-shaped with respect to $p \in S$ if, for each $s \in S$, the line segment $[p, s] \subset S$. $S \subset E$ is said to be star-shaped if it is star-shaped with respect to one of its elements. A convex set is obviously star-shaped.

If $E$ is a normed linear space, $T: S \to S$ is a nonexpansive mapping, if for any pair $x, y \in S$, $\|T(x) - T(y)\| \leqslant \|x - y\|$. $T$ is said to be a contraction if there exists a positive number $k < 1$ such that for any pair $x, y \in S$, $\|T(x) - T(y)\| \leqslant k\,\|x - y\|$. Thus contractive mappings are nonexpansive and any nonexpansive map is continuous.

A mapping $T: S \to S$ is called a Banach operator of type $k$ on $S$, if there exists a constant $k$, $0 \leqslant k < 1$, such that for $x \in S$ $\|T(x) - T^2(x)\| \leqslant k\,\|x - T(x)\|$.

## II. FIXED POINT THEOREMS

In [3] Schauder proved

THEOREM 1. *Any continuous map $T: S \to S$, where $S$ is a compact and convex subset of $E$, has a fixed point.*



0021-9045/89 $3.00

Dotson's well-known fixed point theorem (see [1]) is the following

THEOREM 2. *If $S$ is a compact and star-shaped subset of a normed linear space $E$ and $T: S \to S$ is nonexpansive, then $T$ has a fixed point.*

In [5] Subrahmanyam proved a fixed point theorem for Banach operators:

THEOREM 3. *A continuous Banach operator $T: S \to S$, where $S$ is a closed subset of a Banach space, has a fixed point.*

*Remark.* Theorem 3 remains true if $S$ is a closed subset of a normed linear space and $\mathrm{cl}(T(S))$ is compact.

Using Subrahmanyam's result with the remark we obtain a generalization of Dotson's theorem.

THEOREM 4. *If $:S \to S$ is nonexpansive, where $S$ is a closed and star-shaped subset of a normed linear space $E$, and* $\mathrm{cl}(T(S))$ *is compact, then $T$ has a fixed point.*

*Proof.* Suppose that a subset $S$ of $E$ is star-shaped with respect to $p$. Let us define a sequence of maps $T_n$,

$$T_n(x) = (1 - k_n)\,p + k_n T(x),$$

where $k_n$ is a fixed sequence of positive numbers less than 1 and converging to 1. Each $T_n$ maps $S$ into itself because $T: S \to S$ and $S$ is star-shaped with respect to $p$. Moreover each $T_n$ is a continuous Banach operator of type $k_n$:

$$\begin{aligned} &\|T_n(x) - T_n^2(x)\| \\ &\quad = \|(1-k_n)\,p + k_n T(x) - (1-k_n)\,p \\ &\qquad - k_n T((1-k_n)\,p + k_n T(x))\| \\ &\quad = k_n \,\|T(x) - T((1-k_n)\,p + k_n T(x))\| \\ &\quad \leqslant k_n \,\|x - ((1-k_n)\,p + k_n T(x))\| \\ &\quad = k_n \,\|x - T_n(x)\|. \end{aligned}$$

Since $\mathrm{cl}(T(S))$ is compact, $\mathrm{cl}(T_n(S))$ is compact too, and we may apply the remark: for each $T_n$ there exists a fixed point $x_n$ such that $x_n = T_n(x_n) = (1-k_n)\,p + k_n T(x_n)$. As $\mathrm{cl}(T(S))$ is compact, $\{T(x_n)\}$ has a subsequence $\{T(x_{n_m})\}$ converging, e.g., to $y$; since $k_{n_m} \to 1$, $x_{n_m} = (1-k_{n_m})\,p + k_{n_m} T(x_{n_m})$ converges to $y$. By the continuity of $T$, $T(x_{n_m})$ converges to $T(y)$. But $T(x_{n_m})$ tends to $y$ by the assumption, thus $T(y) = y$, and the proof is complete.

## III. An Application to Invariant Approximations

If $E$ is a normed linear space, $M$ is a subspace of $E$, and $a \in E$, then $d(a, M)$ denotes the distance of point $a$ to set $M$, $d(a, M) = \inf\{\|x-a\|: x \in M\}$, and $O(a, M)$ denotes the set of all best approximations of point $a$ in subspace $M$, $O(a, M) = \{x \in M: \|a-x\| = d(a, M)\}$.

It is well known that set $O(a, M)$ is closed and convex for any subspace $M$ of $E$.

We recall that an operator $A: E \to E$ is compact, if for any bounded subset $S$ of $E$ $\mathrm{cl}(A(S))$ is compact.

If an operator $A: E \to E$ leaves a subspace $M$ of $E$ invariant, then a restriction of $A$ to $M$ will be denoted by the symbol $A \mid M$.

A problem of an approximation theory is existence and uniqueness of best approximation. A partial solution of an existence problem gives the following (see [4])

THEOREM 5. *If $A: E \to E$ is a nonexpansive operator with a fixed point $a$, leaving a subspace $M$ of $E$ invariant, and $A \mid M$ is compact, then the set of best approximations $O(a, M)$ is nonempty.*

Using a fixed point theorem [6, Corollary 2 to Theorem 1], Subrahmanyam obtained the following result [6, Corollary 1 to Theorem 3] which generalizes a theorem of Meinardus:

THEOREM 6. *If $A: E \to E$ is a nonexpansive operator with fixed point $a$, leaving a finite-dimensional subspace $M$ of $E$ invariant, then there exists a best approximation $b \in O(a, M)$ which is also a fixed point of $A$.*

In [4] Smoluk proved a theorem close to the above result; namely, the requirement dealing with the dimension of $M$ is replaced by the assumption that $A$ is linear and $A \mid M$ is compact:

THEOREM 7. *If $A: E \to E$ is a nonexpansive linear operator with fixed point $a$, leaving subspace $M$ of $E$ invariant, and $A \mid M$ is compact, then point $a$ has a best approximation $b$ in $M$ which is also a fixed point of $A$.*

In [4] a question is asked: is it necessary to assume in the theorem above that $A$ is a linear operator? The following theorem shows that this assumption can be abandoned:

THEOREM 8. *If $A: E \to E$ is a nonexpansive operator with fixed point $a$, leaving subspace $M$ of $E$ invariant, and $A \mid M$ is compact, then point $a$ has a best approximation $b$ in $M$, which is also a fixed point of $A$.*

*Proof.* The set $O(a, M)$ is invariant with respect to operator $A$. Indeed, according to Theorem 5 there exists $b \in O(a, M)$, i.e., $d(a, M) = \|a - b\|$. Since $A$ is nonexpansive and $a = A(a)$, we have: $d(a, M) = \|a - b\| \geqslant \|a - A(b)\|$. So $\|a - A(b)\| \leqslant d(a, M)$ for each $b \in O(a, M)$. This means that $A(O(a, M)) \subset O(a, M)$. Let us put $T = A$, $S = O(a, M)$. The set $S$ is closed and star-shaped since it is convex; since $S$ is a bounded subset of $M$ and $T \mid M$ is compact, the set $\mathrm{cl}(T(S))$ is compact. According to Theorem 4 (see part II) operator $T$ has a fixed point in subset $S$; i.e., operator $A$ has a fixed point in the set of best approximations $O(a, M)$, and the proof is complete.

## Acknowledgment

I thank Professor A. Smoluk for proposing this problem and for his many helpful suggestions during the work on this paper.

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